Int. J. Heat Mass Transfer. Vol. 7, pp. 891–900. Pergamon Press 1964. Printed in Great Britain

ENERGY TRANSFER BY RADIATION AND CONDUCTION WITH VARIABLE GAS PROPERTIES

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(*Received* 24 *September* 1963 *and in revised form* 20 *January* 1964)

Abstract—This paper considers the problem of the transfer of energy due to the combined effects of radiation and conduction for a gas with both temperature and frequency dependent properties. The particular problem studied is the one-dimensional energy transfer of an absorbing, emitting and conducting gas. Three approximations for the frequency dependence of the absorption coefficient are considered in detail: (1) the gray gas approximation, (2) the picket fence approximation and (3) a modified picket fence approximation. In addition, the thermal conductivity and the absorption coefficient are assumed to have a power law dependence on the temperature.

NOMENCLATURE

ω,	fractional width of spectral lines;
k,	absorption coefficient;
ηK,	gray background absorption coefficient;
T^*,	temperature;
T_0^*,	reference temperature, taken to be T_w^*;
T,	dimensionless temperature, T^*/T_0^*;
f,	dimensionless free stream temperature;
y,	distance from left wall;
I,	radiation intensity;
q^*,	heat flux;
q,	dimensionless heat flux, $q^*/\sigma T_0^{*4}$;
ν,	radiation frequency;
$E_n(t)$,	exponential integral $= \int_0^1 \mu^{n-2}\, e^{-t/\mu}\, d\mu$;
λ,	thermal conductivity;
τ^*,	optical depth, $\int_0^y k\, dy$;
τ,	$3\tau^*/2$;
$\bar{\tau}$,	$\eta\tau$;
σ,	Stefan–Boltzmann constant;
ϵ,	$\dfrac{3\lambda_0\, k_0}{4\sigma T_0^{*}\ \tau_w^{32}}$;
δ,	$\epsilon\tau_w^2$;
ξ,	$\int_0^y \beta\,[T(y)]\, dy / \int_0^{y_w} \beta\,[T(y)]\, dy$;
$\bar{\xi}$,	$\xi/\epsilon^{1/2}$;
$\Delta\nu_i$,	width of ith spectral line;
ν_i,	radiation frequency at center of ith spectral line.

Superscripts

(0),	constants evaluated as $\delta \to 0$;
$+$,	constants evaluated as $\tau_w \to \infty$;
0,	black-body radiation.

Subscripts

l,	left wall;
w,	right wall;
a,	$= \begin{cases} l \text{ if left wall is being considered} \\ w \text{ if right wall is being considered;} \end{cases}$
ν,	per unit frequency.

INTRODUCTION

HIGH temperature problems require the understanding of the combined effects of radiation, conduction and convection. The study of these interactions has attracted much attention [1–4], and are quite difficult due to the complexity of the basic equations. Lick [5] has proposed approximate analytical techniques for the problem of the transfer of energy due to the combined effects of radiation and conduction which give good agreement with the numerical solution of the original non-linear integro-differential equations.

Two approximations for the absorption coefficient were considered in detail in [5]: (1)

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the gray gas approximation, the absorption coefficient assumed constant, and (2) the picket fence model, the absorption coefficient assumed to consist of an infinite number of spectral lines of uniform height uniformly distributed and superimposed on a gray background. In both cases the absorption coefficient and the thermal conductivity were assumed independent of temperature. The purpose of this investigation is to show that the procedures of [5] may be extended to gases having temperature as well as frequency dependent properties. The thermal conductivity and the absorption coefficient are assumed to have a power law dependence on the temperature. The two cases previously cited for the frequency dependence of the absorption coefficient are considered. In addition, the case when the frequency dependence of the absorption coefficient consists of a finite number of spectral lines superimposed on a gray background is also treated in detail.

ONE-DIMENSIONAL ENERGY FLUX

We consider an absorbing, emitting and conducting gas of variable temperature that is bounded by two infinitely plane parallel walls (see Fig. 1). The walls diffusely emit, absorb, and reflect radiation and are kept at constant temperatures T_1 and T_w with emissivities ϵ_l and ϵ_w.

The equation for the total energy flux due to radiation and conduction is [2, 5]

$$\left.\begin{aligned} q^* = &-\lambda \frac{dT^*}{dy} + 2\pi \int_0^\infty \int_0^{\tau_\nu^*} I_\nu^0 E_2(\tau_\nu^* - t) \\ &dt\, d\nu - 2\pi \int_0^\infty \int_{\tau_\nu^*}^{\tau_{\nu w}^*} I_\nu^0 E_2(t - \tau_\nu^*)\, dt\, d\nu \\ &+ 2\int_0^\infty q_{\nu l}^* E_3(\tau_\nu^*)\, d\nu \\ &- 2\int_0^\infty q_{\nu w}^* E_3(\tau_{\nu w}^* - \tau_\nu^*)\, d\nu \end{aligned}\right\} \quad (1)$$

where I_ν^0 is the black body spectral intensity. The radiative spectral flux leaving the walls is given by

$$\begin{aligned} q_{\nu l}^* &= \epsilon_l q_\nu^0(T_l) + (1 - \epsilon_l) \\ &\times \left[2\pi \int_0^{\tau_{\nu w}^*} I_\nu^0(t) E_2(t)\, dt + 2 q_{\nu w}^* E_3(\tau_{\nu w}^*)\right] \end{aligned} \quad (2)$$

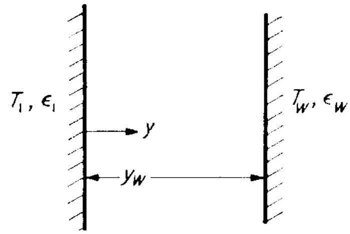

FIG. 1. Diagram of problem.

$$\begin{aligned} q_{\nu w}^* &= \epsilon_w q_\nu^0(T_w) + (1 - \epsilon_w) \\ &\times \left[2\pi \int_0^{\tau_{\nu w}^*} I_\nu^0(t) E_2(\tau_{\nu w}^* - t)\, dt + 2 q_{\nu l}^* E_3(\tau_w^*)\right] \end{aligned} \quad (3)$$

When the absorption coefficient can be approximated by the separable product, $k(\nu, T) = \alpha(\nu)\beta(T)$, equation (1) may be explicitly integrated with respect to the physical length variable, y, yielding [5]

$$\left.\begin{aligned} &\phi(T^*) - \phi(T_l) - \xi\,[\phi(T_w) - \phi(T_l)] = \\ &2\pi \int_0^\infty \int_0^{\tau_{\nu w}^*} \frac{I_\nu^0}{\alpha(\nu)} \{E_3(t)\,[1 - \xi] \\ &- E_3|\tau_\nu^* - t| + \xi E_3(\tau_{\nu w}^* - t)\}\, dt\, d\nu \\ &+ 2\int_0^\infty \frac{q_{\nu l}^*}{\alpha(\nu)} \left\{\frac{1 - \xi}{3} - E_4(\tau_\nu^*)\right. \\ &\left. + \xi E_4(\tau_{\nu w}^*)\right\} d\nu + 2\int_0^\infty \frac{q_{\nu w}^*}{\alpha(\nu)} \\ &\left\{E_4(\tau_{\nu w}^*)\,[1 - \xi] - E_4(\tau_{\nu w}^* - \tau_\nu^*)\right. \\ &\left. + \frac{\xi}{3}\right\} d\nu \end{aligned}\right\} \quad (4)$$

where

$$\phi(T) = \int_0^T \lambda(T)\,\beta(T)\, dT,$$

$$\xi = \int_0^y \beta\,[T(y)]\, dy \Big/ \int_0^{y_w} \beta\,[T(y)]\, dy.$$

Equations (2), (3) and (4) specify the complete problem.

TEMPERATURE DEPENDENT PROPERTIES

Assume the absorption coefficient is given by a power law dependence on the temperature with no dependence on the frequency so that

$$k/k_0 = (T^*/T_0^*)^m \tag{5}$$

and, similarly, for the thermal conductivity

$$\lambda/\lambda_0 = (T^*/T_0^*)^n. \tag{6}$$

In general the actual temperature dependence can be adequately approximated by equations (5) and (6), particularly for moderate temperatures [6, 7, 8, 9].

We note that even the crudest approximations for the absorption coefficient and the thermal conductivity, namely, constants independent of both the frequency and the temperature, require numerical methods for solution. To correct this difficulty, the exact kernel $E_2(t)$ is approximated by the exponential function, $3/4\,e^{-3t/2}$, which has the same area and the same first moment as the exact kernel [5, also see 10, 11]. Making this substitution in equation (4) and using equations (5) and (6), successive differentiations of the resulting equation yield the differential equation,

$$\frac{\epsilon}{m+n+1}\frac{d^2 T^{m+n+1}}{d\xi^2} - \frac{\epsilon\tau_w^2\, T^{m+n+1}}{m+n+1} - T^4 = -\alpha - \beta\tau_w\,\xi \tag{7}$$

where

$$\epsilon = \frac{3}{4}\frac{\lambda_0 k_0}{\sigma T_0^{*3}\tau_w^2} \tag{8}$$

and is a measure of the importance of the heat conducted in comparison to the heat radiated.

The constants α and β are determined by substituting equation (7) into equation (1) to give

$$\left.\begin{aligned}\alpha = \frac{1}{2+\tau_w}\Bigg\{ & q_w + (1+\tau_w)\,q_l \\ & + \frac{\delta}{m+n+1}\Bigg[T_w^{m+n+1} + (1+\tau_w) \\ & T_l^{m+n+1} + \left(\frac{dT^{m+n+1}}{d\tau}\right)_w - (1+\tau_w) \\ & \left(\frac{dT^{m+n+1}}{d\tau}\right)_l\Bigg]\Bigg\}\end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned}\beta = \frac{1}{2+\tau_w}\Bigg\{ & q_w - q_l \\ & + \frac{\delta}{m+n+1}\Bigg[T_w^{m+n+1} - T_l^{m+n+1} \\ & \left(\frac{dT^{m+n+1}}{d\tau}\right)_w + \left(\frac{dT^{m+n+1}}{d\tau}\right)_l\Bigg]\Bigg\}\end{aligned}\right\} \tag{10}$$

and

$$-q = 2\beta \tag{11}$$

where $\delta = \epsilon\tau_w^2$. Thus, α, β and the heat flux q may be directly determined once the derivatives of the temperature and the radiative heat fluxes at the two walls are known.

Boundary-layer analysis

When radiation is the dominant mode of heat transfer, $\epsilon \ll 1$, Equation (7) is of boundary-layer type [12]. Assuming $\tau_w^2 \leqslant 0(1)$, that is $\delta \ll 1$, the free stream solution ($\epsilon = 0$) is

$$T^4 = f^4 = \alpha^{(0)} + \beta^{(0)}\,\tau_w\,\xi \tag{12}$$

where $\alpha^{(0)}$ and $\beta^{(0)}$ are the values of α and β obtained for $\delta = 0$.

In the vicinity of each wall a boundary layer is present due to conduction heat transfer. The boundary-layer equation is obtained by stretching the length variable in equation (7) such that the most highly differentiated term is of the same order of magnitude as the largest terms in the equation. This is obtained with the transformation $\tilde{\xi} = \xi/\epsilon^{1/2}$ and the equation becomes

$$\frac{1}{m+n+1}\frac{d^2 T^{m+n+1}}{d\tilde{\xi}^2} - T^4 = -\alpha^{(0)} - \beta^{(0)}\,\tau_w\,\xi_a = -f_a^4 \tag{13}$$

where the subscript a is l for the boundary layer at the left wall and w for the boundary layer at the right wall. Integrating this equation and matching the free stream and boundary-layer solutions yields the desired derivative

$$\left.\begin{aligned}\delta\left(\frac{dT^{m+n+1}}{d\tau}\right)_a = & \,[2\delta\,(m+n+1)]^{1/2} \\ & \Bigg[\frac{m+n+1}{5+m+n}\left(T_a^{5+m+n} - f_a^{5+m+n}\right) \\ & - f_a^4\left(T_a^{1+m+n} - f_a^{1+m+n}\right)\Bigg]^{1/2}\end{aligned}\right\} \tag{14}$$

Power series expansion

When conduction is dominant, $\epsilon \gg 1$, we approximate the solution by the power series

$$T^{m+n+1} = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 \quad (15)$$

anticipating that the temperature will be a slowly varying function of ξ. The coefficients, a_n, are determined by substituting equation (15) into equation (7).

Diffusion approximation

When the medium has a large optical depth, $\tau_w \gg 1$, equation (7) may be approximated by

$$\frac{\delta T^{m+n+1}}{m+n+1} + T^4 = + \alpha^+ + \beta^+ \tau_w \xi \quad (16)$$

where

$$\alpha^+ = \frac{\delta T_l^{m+n+1}}{m+n+1} + T_l^4 \quad (17)$$

$$\beta^+ = \frac{\delta}{m+n+1} \cdot \frac{T_w^{m+n+1} - T_l^{m+n+1}}{\tau_w} + \frac{T_w^4 - T_l^4}{\tau_w} \quad (18)$$

and

$$-q = \frac{2\delta}{m+n+1} \cdot \frac{(T_w^{m+n+1} - T_l^{m+n+1})}{\tau_w} + 2\frac{T_w^4 - T_l^4)}{\tau_w} \quad (19)$$

These equations represent the diffusion approximation for both radiation and conduction.

Heat flux

The heat flux was calculated for the conditions $\epsilon_l = \epsilon_w = 1{\cdot}0$, $T_l = 0{\cdot}1$, $T_w = 1{\cdot}0$, and $m + n = 1$, by the boundary-layer analysis, power series expansion and diffusion approximation methods. The results are presented in Fig. 2. In addition, the temperature field can be calculated.

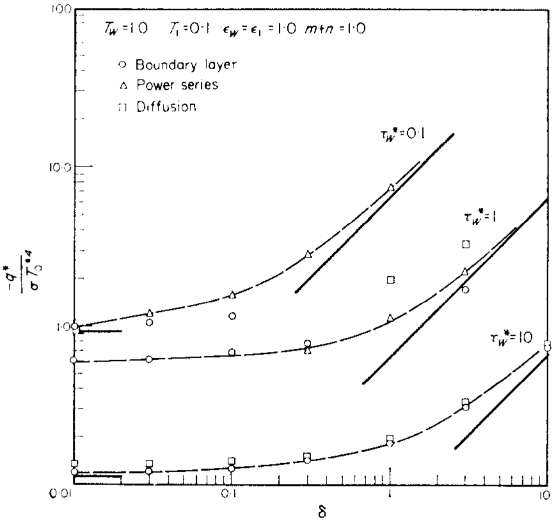

FIG. 2. Heat flux for temperature dependent properties.

TEMPERATURE AND FREQUENCY DEPENDENT PROPERTIES. I

We maintain the same power law dependence on temperature for the absorption coefficient and for the thermal conductivity, that is, equation (5) and (6). In addition, for the frequency dependence we consider a gas with an absorption coefficient consisting of an infinite number of spectral lines superimposed on a uniform gray background. The spectral lines are assumed to have an absorption coefficient k and cover a fraction ω of the spectrum, while the absorption coefficient of the gray background is ηk (see Fig. 3). This model can be considered to be an approximation to the

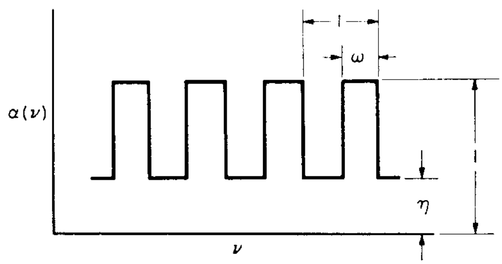

FIG. 3. Frequency dependence of absorption coefficient for picket fence model (I).

absorption of a band spectrum; see, for example [6].†

Differential equation

Making the kernel substitution and taking successive differentiations yields the following differential equation

$$\left.\begin{aligned} &\frac{\epsilon}{m+n+1}\frac{d^4 T^{m+n+1}}{d\xi^4} - \frac{(1+\eta^2)\,\epsilon\,\tau_w^2}{m+n+1} \\ &\quad \frac{d^2 T^{m+n+1}}{d\xi^2} + \frac{\eta^2\,\epsilon\tau_w^4\,T^{m+n+1}}{m+n+1} \\ &= a_1\frac{d^2 T^4}{d\xi^2} - a_2\,\tau_w^2\,T^4 + \tau_w^2\,(\gamma_1 + \gamma_2\,\tau_w\xi) \end{aligned}\right\} \quad (20)$$

$$a_1 = \omega + \eta\,(1-\omega) \quad a_2 = \eta\,(1-\omega) + \eta^2\,\omega$$

The constants γ_1 and γ_2 and the heat flux q may be determined by substituting equation (20) into equation (1). The heat flux is given by

$$-\,q = \frac{2\gamma_2}{\eta^2} \quad (21)$$

Rather than solve for the general result for γ_1 and γ_2 we determine γ_1 and γ_2 for each particular approximate solution by substituting the corresponding solution into equation (1).

Diffusion approximation

When $\tau_w \gg 1$ and $\eta\tau_w \gg 1$, equation (20) may be approximated by

$$\frac{\delta\,\eta^2\,T^{m+n+1}}{m+n+1} + a_2\,T^4 = \gamma_1^+ + \gamma_2^+\,\tau_w\,\xi \quad (22)$$

where

$$\gamma_1^+ = \frac{\eta^2\,\delta\,T_l^{m+n+1}}{m+n+1} + a_2\,T_1^4 \quad (23)$$

$$\gamma_2^+ = \frac{\eta^2\,\delta}{m+n+1}\,\frac{(T_w^{m+n+1} - T_l^{m+n+1})}{\tau_w} + a_2\,\frac{(T_w^4 - T_l^4)}{\tau_w} \quad (24)$$

$$-\,q = \frac{2\delta}{m+n+1}\,\frac{(T_w^{m+n+1} - T_l^{m+n+1})}{\tau_w} + \frac{2a_2}{\eta^2}\,\frac{(T_w^4 - T_l^4)}{\tau_w} \quad (25)$$

† Plass [13] has studied various representations of the absorption coefficient of a band spectrum and made further extensions.

Boundary-layer analysis

When radiation dominates, $\epsilon \ll 1$, the differential equation, equation (20), is of boundary-layer type. Assuming $\tau_w^2 \leqslant 0(1)$, the equation for the free stream temperature variation ($\epsilon = 0$) is

$$a_1\,\frac{d^2 T^4}{d\xi^2} - a_2\,\tau_w^2\,T^4 = -\,\tau_w^2\,(\gamma_1^{(0)} + \gamma_2^{(0)}\,\tau_w\,\xi). \quad (26)$$

Integrating, we obtain

$$T^4 \equiv f^4 = +\frac{\gamma_1^{(0)}}{a_2} + \frac{\gamma_2^{(0)}}{a_2}\,\tau + b_1\,e^{-\mu\tau} + b_2\,e^{+\mu\tau} \quad (27)$$

where $\mu^2 = a_2/a_1$, $\gamma_1^{(0)}$, $\gamma_2^{(0)}$, b_1 and b_2 are found by substituting equation (27) into equation (1).

The boundary-layer equation is

$$\frac{1}{m+n+1}\,\frac{d^4\,T^{m+n+1}}{d\bar{\xi}^4} - a_1\,\frac{d^2\,T^4}{d\bar{\xi}^2} = 0 \quad (28)$$

where $\bar{\xi} = \xi/\epsilon^{1/2}$. Integrating this equation yields

$$\left.\begin{aligned} &\delta\left(\frac{dT^{m+n+1}}{d\tau}\right)_a = [2\,\delta\,a_1\,(m+n+1)]^{1/2}\cdot \\ &\cdot\left[\frac{T_a^{5+m+n} - f_a^{5+m+n}}{(5+m+n)/(1+m+n)} \right. \\ &\left. \quad - f_a^4\,(T_a^{1+m+n} - f_a^{1+m+n})\right]^{1/2} \end{aligned}\right\} \quad (29)$$

The equation for the heat flux is given by

$$\left.\begin{aligned} &-\frac{q}{2}\,[2\,(1+\eta) + \eta\,\tau_w] = \left(a_1 + \frac{a_2}{\eta}\right) \\ &(q_w - q_l) - a_1\,(f_w^4 - f_l^4) \\ &+ \frac{\delta}{m+n+1}\left\{(1+\eta)\left[\left(\frac{dT^{m+n+1}}{d\tau}\right)_w \right.\right. \\ &\left. + \left(\frac{dT^{m+n+1}}{d\tau}\right)_l\right] \\ &\left. + \eta\,(T_w^{m+n+1} - T_l^{m+n+1})\right\} \end{aligned}\right\} \quad (30)$$

Modified boundary-layer analysis

When $\tau_w \gg 1$ but $\eta\,\tau_w \leqslant 0(1)$, the previous approximations are no longer valid. We define a new conduction to radiation parameter, ϵ', where $\epsilon' = \epsilon/a_2 \sim \epsilon/\eta$. Taking the limit of equation (20) as $\tau_w \to \infty$ with ϵ' constant, we obtain

$$\frac{\epsilon'}{m+n+1}\frac{d^2 T^{m+n+1}}{d\xi^2} - \frac{\epsilon' \tilde{\tau}_w^2}{m+n+1} T^{m+n+1} - T^4 = -\frac{\gamma_1}{a_2} - \frac{\gamma_2}{\eta\, a_2}\tilde{\tau}_w\, \xi \quad (31)$$

where $\tilde{\tau}_w = \eta\, \tau_w$. Making the substitutions $\epsilon' \to \epsilon$ and $\tilde{\tau}_w \to \tau_w$ it can be shown that $\gamma_1/a_2 \to \alpha$ and $\gamma_2/\eta\, a_2 \to \beta$ so that equation (31) is equivalent to the equation for the gray gas, equation (7). Therefore, the heat flux is given by

$$-q = \frac{2\gamma_2}{\eta^2} = 2\beta\,(1-\omega) \quad (32)$$

and the results obtained for the gray gas can now be used in the present problem. The above results show that for $\eta \ll 1$, it is the transparency and not the opacity of the gas that is important. The heat flux is primarily due to the part of the spectrum with the small absorption coefficient ηk which comprises the fraction $(1-\omega)$ of the entire spectrum.

Heat flux

The heat flux was calculated for the conditions $\epsilon_l = \epsilon_w = 1{\cdot}0$, $T_l = 0{\cdot}1$, $T_w = 1{\cdot}0$, $\omega = 0{\cdot}5$, $\eta = 0{\cdot}1$, $m + n = 1$. The results are presented in Fig. 4.

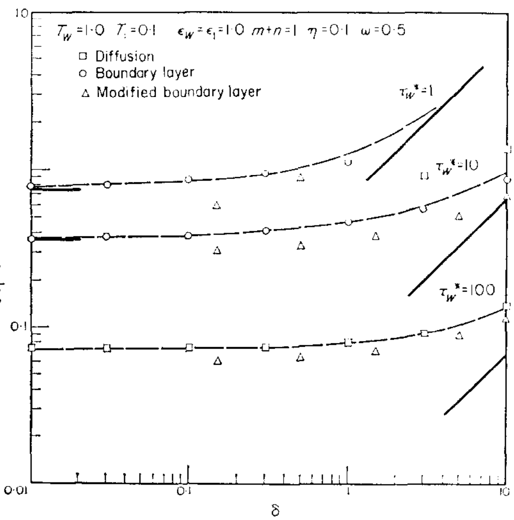

FIG. 4. Heat flux for temperature and frequency dependent properties I.

TEMPERATURE AND FREQUENCY DEPENDENT PROPERTIES. II

We again maintain the same power law dependence on temperature for the absorption coefficient and for the thermal conductivity, equations (5) and (6). However, for the frequency dependence we consider a gas with an absorption coefficient consisting of a finite number of spectral lines superimposed on a uniform gray background. The spectral lines are assumed to have the absorption coefficient k, while the absorption coefficient of the gray background is ηk (see Fig. 5). This representation can be considered to be a model for line absorption or an improved model for band absorption.

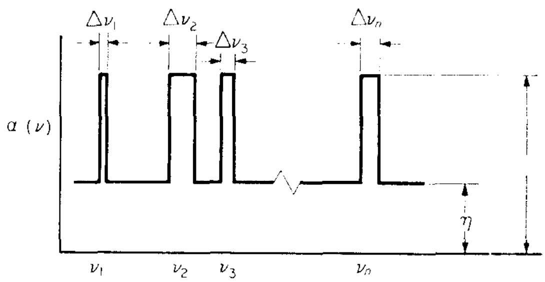

FIG. 5. Frequency dependence of absorption coefficient for modified picket fence model (II).

Differential equation

Making the kernel substitution and taking successive differentiations yields the following differential equation†

$$\left.\begin{aligned}
&\frac{\epsilon}{m+n+1}\frac{d^4 T^{m+n+1}}{d\xi^4} - \frac{(1+\eta^2)}{m+n+1}\\
&\epsilon\,\tau_w^2 \frac{d^2 T^{m+n+1}}{d\xi^2} + \frac{\eta^2\, \epsilon\, \tau_w^4\, T^{m+n+1}}{m+n+1}\\
&= \eta\frac{d^2 T^4}{d\xi^2} - \eta\,\tau_w^2\, T_+^4 + (1-\eta)\frac{d^2}{d\xi^2}\\
&\sum_i \psi_{\nu i}^0 + \eta\,(1-\eta)\,\tau_w^2 \sum_i \psi_{\nu i}^0\\
&+ \tau_w^2\,(\lambda_1 + \lambda_2\, \tau_w\, \xi)
\end{aligned}\right\} \quad (33)$$

† The approximation $\int_{\nu_i - \Delta\nu_i/2}^{\nu_i + \Delta\nu_i/2} I_\nu{}^0\, d\nu \simeq I_{\nu_i}{}^0\, \Delta\nu_i$ has been made.

where

$$\psi^0_{\nu i} = \frac{\pi}{\sigma}\,\frac{2h}{c^2}\,\frac{(\nu_i/T_0^*)^3\cdot(\Delta\nu_i/T_0^*)}{\exp(h\nu_i/kT_0^*T)-1}. \tag{34}$$

The constants λ_1 and λ_2 and the heat flux q may be determined by substituting equation (33) into equation (1). The heat flux is given by

$$-q = \frac{2\lambda_2}{\eta^2}. \tag{35}$$

Rather than solve for the general result for λ_1 and λ_2 we determine λ_1 and λ_2 for each particular approximate solution.

Diffusion approximation

When $\tau_w \gg 1$ and $\eta\,\tau_w \gg 1$, equation (33) may be approximated by

$$\frac{\eta^2\,\delta\,T^{m+n+1}}{m+n+1} + \eta T^4 + \eta\,(\eta-1)\sum_i \psi^0_{\nu i} = \lambda_1^+ + \lambda_2^+\,\tau_w\,\xi \tag{36}$$

where

$$\lambda_1^+ = \frac{\eta^2\,\delta\,T_l^{m+n+1}}{m+n+1} + \eta T_l^4 + \eta\,(\eta-1)\sum_i \psi^0_{\nu i}(T_l) \tag{37}$$

$$\left.\begin{aligned}\lambda_2^+ = {} & \frac{\eta^2\delta}{m+n+1}\,\frac{(T_w^{m+n+1} - T_l^{m+n+1})}{\tau_w} \\ & + \eta\,\frac{(T_w^4 - T_l^4)}{\tau_w} + \frac{\eta\,(\eta-1)}{\tau_w}\,\cdot \\ & \cdot\sum_i\,[\psi^0_{\nu i}(T_w) - \psi^0_{\nu i}(T_l)]\end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned}-q = {} & \frac{2\lambda_2}{\eta^2} = \frac{2\delta}{m+n+1} \\ & \frac{(T_w^{m+n+1} - T_l^{m+n+1})}{\tau_w} + \frac{2}{\eta}\,\frac{(T_w^4 - T_l^4)}{\tau_w} \\ & + 2\,\frac{(\eta-1)}{\eta\,\tau_w}\sum_i\,[\psi^0_{\nu i}(T_w) - \psi^0_{\nu i}(T_l)]\end{aligned}\right\} \tag{39}$$

Boundary-layer analysis

When $\epsilon \ll 1$, equation (33) is of boundary layer type. Assuming $\tau_w^2 \leqslant 0(1)$, the equation for the free stream temperature ($\epsilon = 0$) is

$$\eta\,\frac{\mathrm{d}^2 T^4}{\mathrm{d}\xi^2} - \eta\,\tau_w^2\,T^4 + (1-\eta)\,\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\sum_i \psi^0_{\nu i} + \eta\,(1-\eta)\,\tau_w^2\sum_i \psi^0_{\nu i} = -\lambda_1^{(0)} - \lambda_2^{(0)}\,\tau_w\,\xi \tag{40}$$

Assuming the contributions from the spectral lines to be much smaller than that from the gray background in the region away from the walls, we expand the free stream temperature in powers of the small parameter, Ω, that is

$$T = T_{(0)} + \Omega T_{(1)} + \ldots \tag{41}$$

where

$$\Omega = \frac{\pi}{\sigma}\,\frac{2h}{c^2}\left(\frac{\nu_i}{T_0^*}\right)^3\cdot\frac{\Delta\nu_i}{T_0^*} \tag{42}$$

To zeroth order in Ω we have

$$\eta\,\frac{\mathrm{d}^2 T^4_{(0)}}{\mathrm{d}\xi} - \eta\,\tau_w^2\,T^4_{(0)} = -\,\lambda_1^{(0)} - \lambda_2^{(0)}\,\tau_w\,\xi \tag{43}$$

Integrating, we obtain

$$T^4_{(0)} = \frac{\lambda_1^{(0)}}{\eta} + \frac{\lambda_2^{(0)}}{\eta}\,\tau + c_1\,\mathrm{e}^{-\tau} + c_2\,\mathrm{e}^{+\tau} \tag{44}$$

To this order the effect of the spectral lines is omitted in the region away from the walls. Therefore, the approximation to the free stream temperature given by equation (44) should reduce to the equivalent gray gas result with the absorption coefficient ηk; that is,

$$c_1 = 0,\; c_2 = 0,\; \frac{\lambda_1^{(0)}}{\eta} = \alpha^{(0)} \text{ and } \frac{\lambda_2^{(0)}}{\eta^2} = \beta^{(0)}.$$

This is confirmed by substituting equation (44) into equation (1). For greater accuracy, the higher order terms for the free stream temperature may be determined.

The boundary-layer equation is

$$\frac{1}{m+n+1}\,\frac{\mathrm{d}^4 T^{m+n+1}}{\mathrm{d}\bar{\xi}^4} - \eta\,\frac{\mathrm{d}^2 T^4}{\mathrm{d}\bar{\xi}^2} - (1-\eta)\,\frac{\mathrm{d}^2}{\mathrm{d}\bar{\xi}^2}\sum_i \psi^0_{\nu i} = 0 \tag{45}$$

where $\bar{\xi} = \xi/\epsilon^{1/2}$. Integrating this equation yields

$$\left.\begin{aligned}&\delta\left(\frac{\mathrm{d}T^{m+n+1}}{\mathrm{d}\tau}\right)_a = [2\,\delta\,(m+n+1)]^{1/2}\\&\left\{\eta\cdot\frac{T_a^{5+m+n}-f_a^{5+m+n}}{(5+m+n)/(1+m+n)}\right.\\&+(1-\eta)\int_{f_a^{m+n+1}}^{T_a^{m+n+1}}\sum_i\psi^0_{\nu i}\,\mathrm{d}T^{m+n+1}\\&-\eta f_a^4\,(T_a^{1+m+n}-f_a^{1+m+n})\\&-(1-\eta)\,[\sum_i\psi^0_{\nu t}(f_a)]\\&\left.(T_a^{1+m+n}-f_a^{1+m+n})\right\}^{1/2}\end{aligned}\right\}\quad(46)$$

The equation for the heat flux is given by

$$\left.\begin{aligned}&-\frac{q}{2}[2\,(1+\eta)+\eta\tau_w]=(\eta+1)\,(q_w-q_l)\\&-\eta(f_w^4-f_l^4)-(1-\eta)\\&\sum_i\,[\psi^0_{\nu i}(f_w)-\psi^0_{\nu i}(f_l)]+\frac{\delta}{m+n+1}\\&\left\{(1+\eta)\left[\left(\frac{\mathrm{d}T^{m+n+1}}{\mathrm{d}\tau}\right)_w+\left(\frac{\mathrm{d}T^{m+n+1}}{\mathrm{d}\tau}\right)_l\right]\right.\\&\left.+\,\eta\,(T_w^{m+n+1}-T_l^{m+n+1})\right\}\end{aligned}\right\}\quad(47)$$

Modified boundary-layer analysis

When $\tau_w \gg 1$ but $\eta\ \tau_w \leqslant 0(1)$ the previous approximations are no longer valid. We define a new conduction to radiation parameter, ϵ', with $\epsilon' = \epsilon/\eta$. Taking the limit of equation (33) as $\tau_w \to \infty$ with ϵ' constant, we obtain

$$\frac{\epsilon'}{m+n+1}\,\frac{\mathrm{d}^2\,T^{m+n+1}}{\mathrm{d}\xi^2}-\frac{\epsilon'\,\tilde{\tau}_w^2\,T^{m+n+1}}{m+n+1}-T^4+\sum_i\psi^0_{\nu i}=-\frac{\lambda_1}{\eta}-\frac{\lambda_2}{\eta^2}\tilde{\tau}_w\,\xi\quad(48)$$

where $\tilde{\tau}_w = \eta\ \tau_w$. The solution for the free stream temperature ($\epsilon' = 0$) is

$$f^4-\sum_i\psi^0_{\nu t}(f)=\frac{\lambda_1^{(0)}}{\eta}+\frac{\lambda_2^{(0)}}{\eta^2}\tilde{\tau}\quad(49)$$

The constants $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ are found by substituting equation (49) into equation (1).

The modified boundary layer equation is

$$\frac{1}{m+n+1}\,\frac{\mathrm{d}^2\,T^{m+n+1}}{\mathrm{d}\tilde{\xi}^2}-T^4+\sum_i\psi^0_{\nu i}=-\frac{\lambda_1^{(0)}}{\eta}-\frac{\lambda_2^{(0)}}{\eta^2}\tilde{\tau}\quad(50)$$

where $\tilde{\xi} = \xi/(\epsilon')^{1/2}$. Integrating this equation yields

$$\left.\begin{aligned}&\delta\eta\left(\frac{\mathrm{d}T^{m+n+1}}{\mathrm{d}\tilde{\tau}}\right)_a=[2\,\delta\eta\,(m+n+1)]^{1/2}\\&\left\{\frac{T_a^{5+m+n}-T_a^{5+m+n}}{(5+m+n)/(1+m+n)}\right.\\&-\int_{f_a^{m+n+1}}^{T_a^{m+n+1}}\sum_i\psi^0_{\nu i}\,\mathrm{d}T^{m+n+1}\\&-[f_a^4-\sum_i\psi^0_{\nu i}(f_a)]\\&\left.(T_a^{m+n+1}-f_a^{m+n+1})\right\}^{1/2}\end{aligned}\right\}\quad(51)$$

The equation for the heat flux is given by

$$\left.\begin{aligned}&-\frac{q}{2}\,[2+\tilde{\tau}_w]=(q_w-q_l)-\sum_i\,[\psi^0_{\nu i}(T_w)\\&-\psi^0_{\nu i}(T_l)]+\frac{\delta\eta}{m+n+1}\\&\times\left(\left[\frac{\mathrm{d}T^{m+n+1}}{\mathrm{d}\tilde{\tau}}\right)_w+\left(\frac{\mathrm{d}T^{m+n+1}}{\mathrm{d}\tilde{\tau}}\right)_l\right.\\&\left.+\,T_w^{m+n+1}-T_l^{m+n+1}\right]\end{aligned}\right\}\quad(52)$$

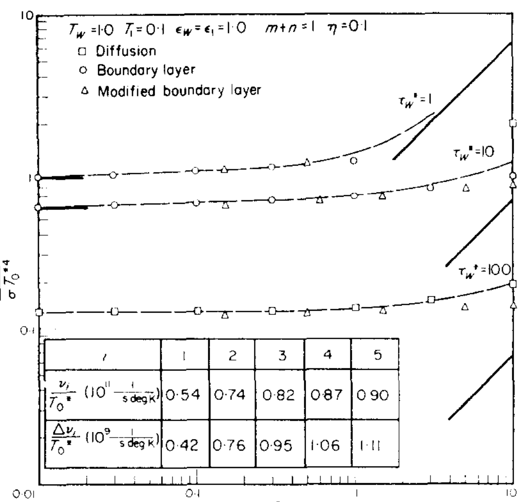

FIG. 6. Heat flux for temperature and frequency dependent properties II.

Heat flux

The heat flux was calculated for the conditions $\epsilon_l = \epsilon_w = 1{\cdot}0$, $T_l = 0{\cdot}1$, $T_w = 1{\cdot}0$, and $m + n + 1$. The values of ν_l/T_0^* and $\Delta\nu_l/T_0^*$ were chosen to correspond to the hydrogen atom at a characteristic shock tube temperature (8400°K) [14, 15, 16]. The results are presented in Fig. 6.

SUMMARY

A study has been made of the transfer of energy due to the combined effects of radiation and conduction for a gas with both temperature and frequency dependent properties. Approximate solutions have been obtained for the heat flux and the temperature field. The following frequency dependent models of the absorption coefficient were considered: (1) The gray gas model, (2) The picket fence model, and (3) a modified picket fence model. In all cases the thermal conductivity and the absorption coefficient were assumed to have a power law dependence on the temperature.

ACKNOWLEDGEMENT

The author is indebted to Professor Wilbert Lick of Harvard University for many helpful discussions and suggestions. The research was supported in part by National Science Foundation Grant 7588.

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Résumé—Cet article considère le problème du transport d'énergie dû aux effets combinés du rayonnement et de la conduction pour un gaz avec des propriétés dépendant à la fois de la température et de la fréquence. Le problème particulier étudié est le transport d'énergie unidimensionnel d'un gaz absorbant, émetteur et conducteur. Trois approximations pour la dépendance en fréquence du coefficient d'absorption sont examinées en détail: (1) l'approximation du gaz gris, (2) l'approximation de la palissade de piquets, (3) une approximation modifiée de la palissade de piquets. En plus, on a supposé que la conductivité thermique et le coefficient d'absorption dépendent de la température selon une loi en puissance.

Zusammenfassung—In der Arbeit wird der Energietransport bei gleichzeitiger Strahlung und Leitung in einem Gas mit temperatur- und frequenzabhängigen Stoffwerten untersucht. Speziell wird der eindimensionale Energietransport in einem absorbierenden, strahlenden und leitenden Gas betrachtet. Für die Frequenzabhängigkeit des Absorptionskoeffizienten werden drei Näherungen im einzelnen untersucht: (1) die Näherung für graues Gas; (2) die Lattenzaunnäherung und (3) eine abgewandelte Lattenzaunnäherung. Zusätzlich wird angenommen, dass die thermische Leitfähigkeit und der , bsorptionskoeffizient von der Temperatur nach einem Potenzgesetz abhängt.

Аннотация—В статье рассматриваются вопросы переноса энергии в результате совместного действия излучения и теплопроводности при свойствах газов, зависящих от температуры и частоты. Рассмотрена частная задача одномерного переноса энергии для поглощающих, излучающих и проводящих сред. Подробно рассмотрены три приближения для зависимости частоты от коэффициента поглощения: 1. приближение серого газа; 2. ступенчатое приближение; 3. модифицированное ступенчатое приближение. Кроме того, принят степенной закон зависимости коэффициентов теплопроводности и поглощения от температуры.